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Assembly procedure for elementary matrices of train-track-bridge railway system

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Abstract. The study of railway dynamics remains an active and fertile field of research, given the technological evolution of this transport system. Several modeling methods have been developed to explore the dynamic interactions of a train-rail-bridge system, and these numerical models enable optimization of bridge design, especially for high-speed lines. In this perspective, this work joins this huge modeling project by proposing a procedure for assembling the elementary matrices of a dynamic system composed of a train, a rail and a bridge, in order to obtain the global differential equations of the system. The model studied consists of a moving part, the vehicle, modeled by a mass-spring-damper system, and a fixed part, the rail and bridge deck, modeled by two Bernoulli beams. The elements to be assembled are not identical, which increases the complexity of their assembly, as the position of the vehicle wheel changes as a function of time and passes from one element to another, creating loaded and unloaded elements.

The dynamic equations were solved using the Newmark Beta numerical method. The model developed was subjected to a validation process based on a comparison of the dynamic responses of the bridge and vehicle obtained by the present study and those presented by previous studies.

AMS Subject Classification (2020): 37N30

Keywords: Dynamic System, Elementary Matrices, Finite Element method, Process Assembling, Newmark Beta Integration Method

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1. Introduction and Preliminaries

Numerical simulation has become an effective tool for dynamic prediction of vehicle, rail and bridge behavior in the railway field [8]. Several models have been developed in various studies, including [3], [7], [14], which presented the vehicle-rail-bridge in two subsystems namely the vehicle and the bridge-rail. The vehicle-rail-bridge system was considered as a single coupled system, the dynamic contact forces between train and rail being internal forces [6], [15].

Dynamic responses determination of a railway system involves 3 critical steps requiring high accuracy respectively:

- Modeling and elaboration of elementary differential equations,
- Assembling elementary matrices to establish equations of overall system,
- Selecting a numerical method for solving equations, and consequently for predicting the dynamic behavior of the system studied.

Several methods have been used to determine motion equations, Lou and Zeng [9] reported the motion equations of the vehicle-rail-bridge system using Hamilton's principle. D'Alembert's principle was used [20] to establish set of differential equations.

The principle of the stationary value of total potential energy combined with the finite element method has been adopted in current study to establish motion equations of the system.

The dynamic interaction model of vehicle-rail-bridge system has evolved over time from a simple restricted model of constant force in motion to axle loads. Several studies have adopted the restricted model [2], [13], which sought only to predict axle response without taking accounttrack irregularities. The moving and suspended mass model remained simple after moving load and allowed us to study the effect of vehicle inertia on axle response [1], [19].

Other researchers have modeled train by two rigid masses connected by spring-damper suspension systems [4], [18]. The moving systems provide more realistic modeling and behavior for the interaction issue. The system includes a first mass which represents the vehicle body and the second represents the wheel mass. This model has 2 DOFs (degrees of freedom) respectively the vertical displacements of body and the wheel. It has been modified to incorporate the effect of pitching and become a 4 DOFs model [17]. The 4 DOFs are the two wheels vertical displacements, the body vertical displacement and the body rotation.

[16] developed a 10 DOFs model, the train being modeled by a body, two bogies and 4 wheels, connected respectively by a primary and a secondary suspension represented by spring-damper systems. This model has 10 DOFs and allows the study of bridge dynamic response and passengers comfort [10], [12]. [21] presented a more complex 115 DOFs train model including vertical and transverse connections between car bodies and two suspension layers.

However, the preceding models do not present a suitable assembly process enabling the transition from elementary matrices to global matrices. This article therefore presents a detailed assembly procedure for the elementary matrices of a vehicle-train-bridge railway system.

The particularity of the procedure developed is that it allows the assembly of non-identical matrices composed of discrete masses representing the system of vehicle and continuous masses representing the system of rail and the bridge. The ballast is modeled by damping springs, and its mass is integrated into the mass of the bridge, which is assumed to be simply supported. The two systems are coupled, and contact forces are considered as internal forces.

The rail-bridge sub-structure is divided into 10 elements of length l. The elementary matrices of the sub-system are established using the finite element method combined with the potential energy principle. The proposed assembly method consists of assembling the mass, stiffness, damping and force vector matrices of the sub-structure without any vehicle action.

Once the vehicle's position has been determined, the elements representing wheel-rail interaction will be introduced into the substructure matrices. By adding the vehicle elements, we obtain the global matrices of the system under study.

The numerical solutions of the motion equations are obtained by direct step-by-step integration in the time domain using the Newmark Beta method. The results are presented in Figures 6, 8 and 10, and show excellent agreement when compared with the dynamic responses calculated by the modal analysis method presented in [1], Figures 5, 7 and 9

2. Equations of motion for "Tarin-track-bridge" interaction

2.1. The theoretical model

The vehicle rail and bridge are considered a complete system in current investigation. The vehicle is modeled by two rigid respective masses, m_1 body mass and m_e wheel mass. The two are connected by a spring-damper system (k_1, c_1), q_1 and q_e present vertical displacements of the vehicle which is running with a velocity v(t) and an acceleration a(t) in longitudinal direction. Therefore, total number of DOFs is two, since vertical movement of wheel is limited by rail, the number of independent DOFs becomes 1 noted $\{q_1\}$.

Rail and bridge are modeled by two Euler-Bernoulli beams of finite length simply supported on bridge piers. The two are connected by a continuous layer of damper springs (k_{rp}, c_{rp}).

Based on the finite element method, rail and bridge are divided into 10 elements of length l, the damping of the rail is neglected, however a linear viscous damping c_p is considered for the bridge, m_b and m_r represent respectively the mass per unit length for bridge and rail, E_r and I_r represent Young's modulus and constant moment of inertia of rail, E_p and I_p represent Young's modulus and constant moment of inertia of bridge.

When longitudinal displacement is neglected of the two beams, each beam element has 4 DOFs, vertical displacement and rotation in every extremity. Displacement vectors respectively for rail and bridge are $\{q_2, q_3, q_4, q_5\}$ and $\{q_6, q_7, q_8, q_9\}$. Therefore vehicle-rail-bridge element become 9 DOFs $\{q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9\}$.

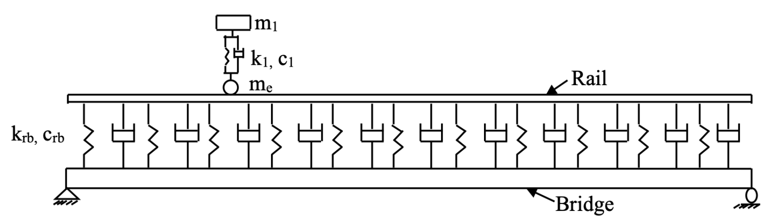

Figure 1: Model of Vehicle-Track-Bridge Interaction System

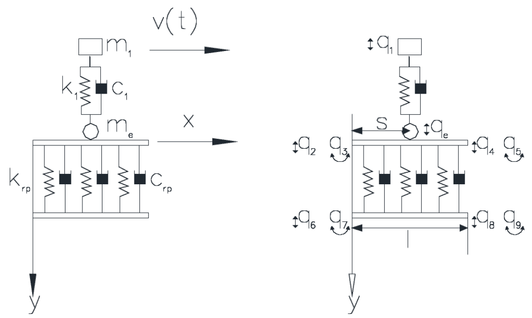

Figure 2: Model of Vehicle-Track-Bridge Interaction Element

At a fixed time t, the wheel of mass m_e is located at a distance s from rail left end. We note that: $x{=}v \times t$ and j is the number of the element on which the wheel acts, with $j = E(\frac{x}{l}) + 1$, we specify that $E(\frac{x}{l})$ the integer part of the number, v is the wheel speed. In this case $s = x - (j - 1) \times l$.

Taking into consideration track irregularities causing vertical deviation along the rail from its initially horizontal profile, let r(s) be the value of this deviation at the point of contact with wheel.

We note that $N\left(s\right)$ represents the cubic hermite interplation functions and $N\left(s\right)^T$ represents the transpose of $N\left(s\right)$ where

$$N(s) = [N_A N_B N_C N_D] \tag{1}$$

$$N_A = 1 - 3\left(\frac{s}{l}\right)^2 + 2\left(\frac{s}{l}\right)^3 \tag{2}$$

$$N_B = s\left[1 - 2\left(\frac{s}{l}\right) + \left(\frac{s}{l}\right)^2\right] \tag{3}$$

$$N_C{=}3\left(\frac{s}{l}\right)^2 - 2\left(\frac{s}{l}\right)^3 \tag{4}$$

$$N_D{=}\ s\left[\left(\frac{s}{l}\right)^2 - \left(\frac{s}{l}\right)\right] \tag{5}$$

2.2 Formulation of motion equations

Adopting the method based on total potential energy principle of the stationary value combined with the finite element method, described in detail in the Lou and al. [1], we obtain the motion equations of the vehicle-rail bridge element in matrix form as follows:

$$M_e\,Q_e+C_e\,\dot{Q}_e+K_e\,\ddot{Q}_e=F_e \tag{6}$$

with:

$$Me=\begin{bmatrix}[m_{11}] & [0] & [0]\\ \{0\} & [m_{rr}] & [0]\\ \{0\} & [0] & [m_{pp}]\end{bmatrix}$$

$$Ke=\begin{bmatrix}[k_{11}] & [k_{1r}] & [0]\\ \{k_{r1}\} & [k_{rr}] & [k_{rp}]\\ \{0\} & [k_{pr}] & [k_{pp}]\end{bmatrix}$$

$$Ce=\begin{bmatrix}[c_{11}] & [c_{1r}] & [0]\\ \{c_{r1}\} & [c_{rr}] & [c_{rp}]\\ \{0\} & [c_{pr}] & [c_{pp}]\end{bmatrix}$$

$$\ddot{Q}_e=\begin{Bmatrix}\ddot{q}_1\\ \{\ddot{q}_r\}\\ \{\ddot{q}_p\}\end{Bmatrix},\ \dot{Q}_e=\begin{Bmatrix}\dot{q}_1\\ \{\dot{q}_r\}\\ \{\dot{q}_p\}\end{Bmatrix}$$

$$Q_e=\begin{Bmatrix}q_1\\ \{q_r\}\\ \{q_p\}\end{Bmatrix},\ F_e=\begin{Bmatrix}f_1\\ \{f_r\}\\ \{f_p\}\end{Bmatrix}$$

Vector of nodal rail displacement:

$$\{q_r\}=\begin{Bmatrix}q_2\\ q_3\\ q_4\\ q_5\end{Bmatrix}$$

Vector of nodal bridge displacement:

$$\{q_p\}=\begin{Bmatrix}q_6\\ q_7\\ q_8\\ q_9\end{Bmatrix}$$

Mass Matrix component:

$$[m_{11}]=m_1$$

$$[m_{rr}]=m_e N(s)^T N(s)+m_r \int_0^l N(x)^T N(x)\,dx$$

$$[m_{pp}]=m_p \int_0^l N(x)^T N(x)\,dx$$

Damping Matrix component:

$$[c_{11}]=c_1$$

$$\{c_{r1}\}=\{c_{1r}\}=-c_1 N(s)$$

$$[c_{rp}]=[c_{pr}]=-c_{rp}\int_0^l N(x)^T N(x)\,dx$$

$$[c_{pp}]=(c_{rp}+c_p)\int_0^l N(x)^T N(x)\,dx$$

$$[c_{rr}]=c_{rp}\int_0^l N(x)^T N(x)\,dx+c_1 N(s)^T N(s)+2v m_e N(s)^T N'(s)$$

Stiffness Matrix component:

$$[k_{11}]=k_1$$

$$[k_{1r}]=-k_1 N(s)-c_1 v N'(s)$$

$$\{k_{r1}\}=-k_1 N(s)^T$$

$$[k_{rp}]=[k_{pr}]=-k_{rp}\int_0^l N(x)^T N(x)\,dx$$

$$[k_{pp}]=E_p I_p \int_0^l N''(x)^T N''(x)\,dx+k_{rp}\int_0^l N(x)^T N(x)\,dx$$

$$[k_{rr}]=E_r I_r \int_0^l N''(x)^T N''(x)\,dx+k_{rp}\int_0^l N(x)^T N(x)\,dx+k_1 N(s)^T N(s)$$
$$+(c_1 v+m_e a)N(s)^T N'(s)+m_e v^2 N(s)^T N''(s)$$

Force Vector components:

$$f_1=k_1 r(s)+c_1 v r'(s)$$

$$\{f_r\}=[(m_1+m_e)\,g-k_1 r(s)-c_1 v r'(s)-m_e(a r'(s)+v^2 r''(s))]N(s)^T$$

$$\{f_p\}=\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

We note that:

$$[A]=\int_0^l N(x)^T N(x)\,dx = \begin{bmatrix} \frac{13l}{35} & \frac{11l^2}{210} & \frac{9l}{70} & \frac{-13l^2}{420} \\ \frac{11l^2}{210} & \frac{l^3}{105} & \frac{13l^2}{420} & -\frac{l^3}{140} \\ \frac{9l}{70} & \frac{13l^2}{420} & \frac{13l}{35} & -\frac{11l^2}{210} \\ \frac{-13l^2}{420} & -\frac{l^3}{140} & \frac{11l^2}{210} & \frac{l^3}{105} \end{bmatrix}$$

$$[B]=\int_0^l N''(x)^T N''(x)\,dx = \begin{bmatrix} \frac{12}{l^3} & \frac{6}{l^2} & \frac{-12}{l^3} & \frac{6}{l^2} \\ \frac{6}{l^2} & \frac{4}{l} & -\frac{6}{l^2} & \frac{2}{l} \\ \frac{-12}{l^3} & \frac{6}{l^2} & \frac{12}{l^3} & -\frac{6}{l^2} \\ \frac{6}{l^2} & \frac{2}{l} & \frac{-6}{l^2} & \frac{4}{l} \end{bmatrix}$$

2.3. Proposed assembly procedure

The proposed method consists of assembling stiffness, mass and damping matrices, as the force vector of the system comprising rail and bridge, without any vehicle action.

2.3.1. Bridge-rail elementary interaction matrix

To establish the elementary matrices of the rail-bridge element without any railway load, one proceeds as follows:

- Eliminate the first row and column of each elementary matrix M_e, K_e, C_e defined above

- Eliminate the first component of each vector Q_e $\dot{Q}_e$, $\ddot{Q}_e$ and force vector F_e

- Annul the elements defining the rail load: $m_1 = m_e = 0$, $k_1 = 0$ and $c_1 = 0$

The mass, stiffness and damping matrices as well as the force vector are defining interactions of bridge-rail system for a single element:

Elementary Mass Matrix $[M_{rbe}]$

$$[M_{rbe}] = \begin{bmatrix} [m_{rr}'] & [0] \\ [0] & [m_{pp}] \end{bmatrix}$$

$[m_{rr'}]$ and $[m_{pp}]$ blocks are defined as follows

$$[m_{rr'}] = m_r\,[A]$$

$$[m_{pp}] = m_p\,[A]$$

Elementary Stiffness Matrix $[K_{rbe}]$

$$[K_{rbe}] = \begin{bmatrix} [k_{rr}'] & [k_{rp}] \\ [k_{pr}] & [k_{pp}] \end{bmatrix}$$

$[k_{rr'}]$, $[k_{rp}]$ and $[k_{pp}]$ blocks are defined as follows:

$$[k_{rp}] = [k_{pr}] = -k_{rp}\,[A]$$

$$[k_{rr'}] = E_r I_r\,[B] + k_{rp}\,[A]$$

$$[k_{pp}] = E_p I_p\,[B] + k_{rp}\,[A]$$

Elementary Damping Matrix $[C_{rbe}]$

$$[C_{rbe}] = \begin{bmatrix} [c_{rr}'] & [c_{rp}] \\ [c_{pr}] & [c_{pp}] \end{bmatrix}$$

$[c_{rr'}]$, $[c_{pr}]$ and $[c_{pp}]$ blocks are defined as follows:

$$[c_{rr'}] = c_{rp}\,[A]$$

$$[c_{rp}] = [c_{pr}] = -c_{rp}\,[A]$$

$$[c_{pp}] = (c_{rp} + c_p)\,[A]$$

Elementary Force Vector:

$$\{f_{rbe}\} = \begin{Bmatrix} 0 \\ . \\ . \\ 0 \end{Bmatrix}$$

the matrices $[M_{rbe}]$,$[K_{rbe}]$ and $[C_{rbe}]$ are square matrices of size 8×8 the force vector $\{f_{rbe}\}$ is of size 8×1

2.3.2. Assembly of elementary matrix into global bridge-rail system matrix

Bridge and rail are decomposed into 10 elements (E) of equal length l, corresponding to 22 nodes. Each node has 2 DOFs refer to figure 3, resulting in a total of 44 DOFs:

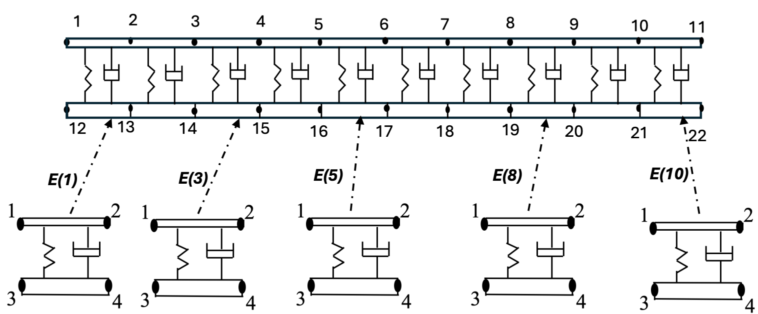

Figure 3: Rail-Bridge structure meshing

The mesh is defined by the following connectivity:

Table 1: Connectivity table

Elements	Elements nodes
1	[1 2 12 13]
2	[2 3 13 14]
3	[3 4 14 15]
4	[4 5 15 16]
5	[5 6 16 17]
6	[6 7 17 18]
7	[7 8 18 19]
8	[8 9 19 20]
9	[9 10 20 21]
10	[10 11 21 22]

The assembly technique requires the elementary localization table of the degrees of freedom associated with each mesh finite element. This table is composed of columns corresponding to the elements connectivity vectors, and rows representing the mesh element degrees of freedom.

Table 2(A): Localization of degrees of freedom

i	$CV_1(i)$	$CV_2(i)$	$CV_3(i)$	$CV_4(i)$	$CV_5(i)$	$CV_6(i)$
1	1	3	5	7	9	11
2	2	4	6	8	10	12
3	3	5	7	9	11	13
4	4	6	8	10	12	14
5	23	25	27	29	31	33
6	24	26	28	30	32	34
7	25	27	29	31	33	35
8	26	28	30	32	34	36

Table 2(B): Localization of degrees of freedom

i	$CV_7(i)$	$CV_8(i)$	$CV_9(i)$	$CV_{10}(i)$
1	13	15	17	19
2	14	16	18	20
3	15	17	19	21
4	16	18	20	22
5	35	37	39	41
6	36	38	40	42
7	37	39	41	43
8	38	40	42	44

The assembly function is defined as follows:

For all $(i, j) \in \begin{bmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\end{bmatrix}^2$

$$\begin{cases} G\left(CV_p(i), CV_p(j)\right) = g(i,j) & p{=}1 \\ G\left(CV_p(i), CV_p(j)\right) = G\left(CV_p(i), CV_p(j)\right) + g(i,j) & p \neq 1 \end{cases}$$

While:

- g is the elementary matrix to be assembled and g(i,j) is the element of the ith row and jth column
- G is the global matrix of the system and G(I, J) is the element of the Ith row and the Jth column, such that : I= CVp(i) , J= CVp(j)
- CV_p is the connectivity vector of the element p

Global matrices of the bridge-rail system $[C_{rbg}]$, $[K_{rbg}]$ and $[M_{rbg}]$ are obtained respectively by assembling the elementary matrices $[C_{rbe}]$, $[K_{rbe}]$, $[M_{rbe}]$ using the function defined above.

$\{f_{rbg}\}$ represents the force vector of the rail-bridge system where all elements are zero.

The global mass, stiffness and damping matrix are 44 x 44 in size, and global displacement vector and force vector are 44 x 1 in size.

2.3.3. Matrix of "vehicle-rail" interaction

$N_g(s)$ is of size 1×44 and all its elements are null except those which correspond to rail element DOF on which the wheel operates

$$N_g(s) = \begin{bmatrix} 0 & 0 & 0 & N_A & N_B & N_c & N_D & 0 & 0 & 0 \end{bmatrix}$$

Mass matrix. Consider $[M_{rv}]$ the mass matrix induced by the moving mass on the rail, with size 44 x 44, s denotes the local coordinate measured from the left node of a beam element.

$$[M_{rv}] = m_e \times N_g(s)^T \times N_g(s)$$

Stiffness matrix. Consider $[K_{rv}]$, $[k_{1r}]$ and $\{k_{r1}\}$ are the stiffness matrices induced by wheel displacement on rail, they are respectively of order 44×44, 1×44 and 44×1 with:

$$[K_{rv}] = k_1 N_g(s)^T N_g(s) + (c_1 v + m_e a) N_g(s)^T N'_g(s) + m_e v^2 N_g(s)^T N''_g(s)$$

$$[k_{1r}] = -k_1 N_g(s) - c_1 v N'_g(s)$$

$$\{k_{r1}\} = -k_1 N_g(s)^T$$

Damping matrix. The matrices $[C_{rv}]$, $[c_{1r}]$ and $\{c_{r1}\}$ are of orders 44×44 ,1×44 and 44×1 respectively and represent overall damping matrices induced by wheel on rail.

$$[C_{rv}] = c_1 N_g(s)^T N(s) + 2 v m_e N_g(s)^T N'_g(s)$$

$$[c_{1r}] = -c_1 N_g(s)$$

$$\{c_{r1}\} = -c_1 N_g(s)$$

Force vector. $\{f_{rv}\}$ represent the load force vector induced by the wheel acting on the rail, it is of order 44×1:

$$\{f_{rv}\} = [(m_1 + m_e) g - k_1 r(s) - c_1 v r'(s) - m_e \left(a r'(s) + v^2 r''(s)\right)] N_g(s)^T$$

2.3.4. Construction of global matrices for vehicle-rail-bridge

Global matrices of the system composed of 3 elements: vehicle, rail and bridge are written as follows:

Mass Matrix:

$$M_g = \begin{bmatrix} [m_{11}] & [0] \\ \{0\} & [M_{rbg}] + [M_{rv}] \end{bmatrix}$$

Damping Matrix:

$$C_g = \begin{bmatrix} [c_{11}] & [c_{1r}] \\ \{c_{r1}\} & [C_{rbg}] + [C_{rv}] \end{bmatrix}$$

Stiffness Matrix:

$$K_g = \begin{bmatrix} [k_{11}] & [k_{1r}] \\ \{k_{r1}\} & [K_{rbg}] + [K_{rv}] \end{bmatrix}$$

Force Vector:

$$F_g = \begin{Bmatrix} f_1 \\ \{f_{rv}\} + \{f_{rbg}\} \end{Bmatrix}$$

The dynamic response vectors are respectively:

Displacement:

$$Q_g = \begin{Bmatrix} q_1 \\ \{q_r\} \\ \{q_p\} \end{Bmatrix}$$

Velocity:

$$\dot{Q}_g = \begin{Bmatrix} \dot{q}_1 \\ \{\dot{q}_r\} \\ \{\dot{q}_p\} \end{Bmatrix}$$

Acceleration:

$$\ddot{Q}_g = \begin{Bmatrix} \ddot{q}_1 \\ \{\ddot{q}_r\} \\ \{\ddot{q}_p\} \end{Bmatrix}$$

It should be noted that:

- $\{\ddot{q}_r\}$, $\{\dot{q}_r\}$ and $\{q_r\}$ are of order 22×1. (The rail's degrees of freedom)

- $\{\ddot{q}_p\}$, $\{\dot{q}_p\}$ and $\{q_p\}$ are of order 22×1. (The bridge's degrees of freedom)

- $\{f_{rv}\}$ is of order 44×1

- $\{f_p\}$ is a null vector of ordre 44×1

The assembly process used allows genering the global motion equation for the entire vehicle-rail-bridge interaction system follows:

$$M_g\,\ddot{Q}_g + K_g\,Q_g + C_g\,\dot{Q}_g = F_g \tag{7}$$

3. Dynamic analysis using Newmark beta method

Several numerical integration methods in the time domain [5] have been reported to solve this type of equation. Three main requirements must be addressed in a numerical solution procedure, namely:

- Convergence: numerical solution approaches the exact solution as the time step decreases.

- Accuracy: the numerical solution presents result close enough to exact solution.

- Stability: the solution must be stable even in presence of errors.

The Newmark beta method remains one of the most popular explicit methods used in structures dynamic analysis. It is applicable to linear differential systems with time-dependent mass, stiffness and damping matrices [11].

In addition to β parameter, it contains a δ parameter with the value $\frac{1}{2}$. The Newmark equation can be written in incremental quantities for a constant time step Δt as follows:

$$\Delta u_i = \dot{u}_i \Delta t + \frac{1}{2} \ddot{u}_i \Delta t^2 + \beta \Delta \ddot{u}_i \Delta t^2 \quad (8)$$

$$\Delta \dot{u}_i = \ddot{u}_i \Delta t + \delta \Delta \ddot{u}_i \Delta t \quad (9)$$

$$\Delta u = u\,(t + \Delta t) - u\,(t) \quad (10)$$

$$\Delta \dot{u} = \dot{u}\,(t + \Delta t) - \dot{u}\,(t) \quad (11)$$

$$\Delta \ddot{u} = \left(\Delta u - \dot{u}_t \Delta t - \frac{1}{2} \ddot{u}_t \Delta t^2 \right) / \beta \Delta t^2 \quad (12)$$

Substituting (12) into (9)

$$\Delta \dot{u} = \frac{1}{2\beta \Delta t} \Delta u - \frac{1}{2\beta} \dot{u}_t + \left(1 - \frac{1}{4\beta} \right) \ddot{u}_t \Delta t^2 \quad (13)$$

Considering the equation of motion at time t and time $t + \Delta t$.

$$M \ddot{u}_t + C \dot{u}_t + K u_t = F_t \quad (14)$$

$$M \ddot{u}_{t+\Delta t} + C \dot{u}_{t+\Delta t} + K u_{t+\Delta t} = F_{t+\Delta t} \quad (15)$$

Subtracting (14) from (15):

$$M \Delta \ddot{u} + C \Delta \dot{u} + K \Delta u = \Delta F \quad (16)$$

with

$$\Delta \ddot{u} = \ddot{u}_{t+\Delta t} - \ddot{u}_t \quad (17)$$

$$\Delta \dot{u} = \dot{u}_{t+\Delta t} - \dot{u}_t \quad (18)$$

$$\Delta u = u_{t+\Delta t} - u_t \quad (19)$$

The values of $M, CandK$ in (16) are calculated at time t, assuming that remain constant during Δt.

Substituting (12) and (13) into (16) give (20), which calculates displacement Δu:

$$\widehat{K_t} \Delta u = \widehat{\Delta F} \quad (20)$$

where the effective stiffness $\widehat{K_t}$ and incremental force $\widehat{\Delta F}$ are provided respectively

$$\widehat{K_t} = K_t + \frac{1}{\beta \Delta t^2} M_t + \frac{1}{2\beta \Delta t} C_t \qquad (21)$$

$$\widehat{\Delta F} = \Delta F + \frac{1}{\beta \Delta t} M_t \dot{u}_t + \frac{1}{2\beta} C_t \dot{u}_t + \frac{1}{2\beta} M_t \ddot{u}_t - C_t \Delta t \left(1 - \frac{1}{4\beta}\right) \ddot{u}_t \qquad (22)$$

The value of β parameter range between 1/6 and 1/2.

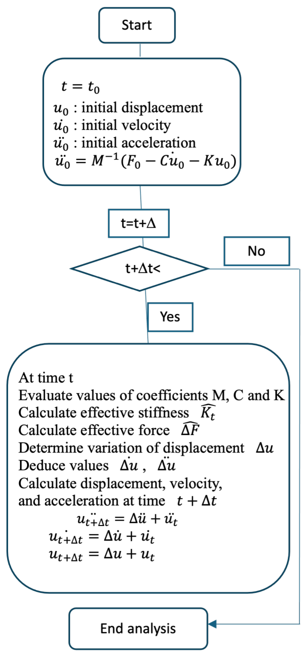

Figure 4: Flowchart for equation resolution process

4. Validation of proposed procedure

The assembly procedure used to obtain motion equations of vehicle-rail-bridge interaction system and the associated computer program are verified through the study in this section. Differential equations were solved by Newmark-β method with a time step $\Delta t = 0.005$. The computational program was developed on "MATLAB © R2021b, The MathWorks, Inc."

The example investigated to validate the proposed procedure consists of a simple supported Bernoulli beam of length L = 30 m, travelling with a single-axle vehicle at constant speed v = 27.78 m/s. The beam surface is assumed to be smooth $r\,(s) = 0$. Vehicle characteristics, bridge and rail are summarized in Table 2 [8].

Table 2: System Vehicle-Rail-Bridge Parameters

Parameters	Value	Unit
Vehicle		
m_1	*5750*	Kg
m_e	*0*	Kg
c_1	*0*	Ns/m
k_1	$1.595\ 10^{6}$	N/m
Rail		
m_r	10^{-7}	Kg/m
I_r	10^{-10}	m^4
E_r	$2.06\ 10^{11}$	Pa
k_{rp}	10^{13}	N/m
c_{rp}	*0*	Ns/m
Bridge		
m_b	$2.303\ 10^{3}$	Kg/m
I_b	*2.90*	m^4
E_b	$2.87\ 10^{9}$	Pa
c_p	*0*	Ns/m

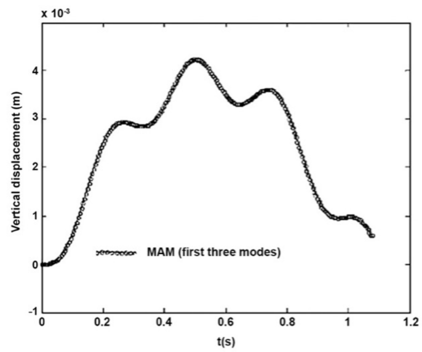

Figure 5: Time history of vertical displacement of midpoint of bridge computed by modal analysis method [8]

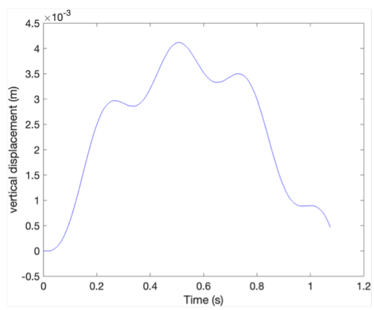

Figure 6: Time history of vertical displacement of midpoint of bridge calculated by current method

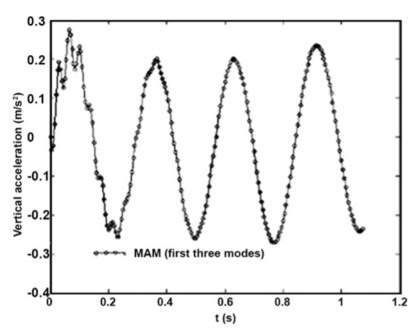

Figure 7: Time history of vertical acceleration of midpoint of bridge computed by analysis method [8]

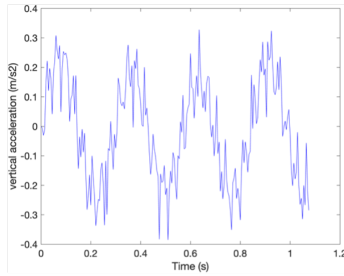

Figure 8: Time history of vertical acceleration of midpoint of bridge calculated by current method

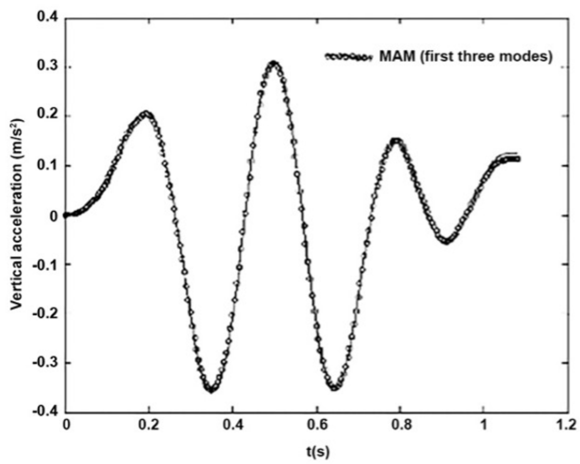

Figure 9: Time history of vertical acceleration of carbody computed by modal analysis method [1]

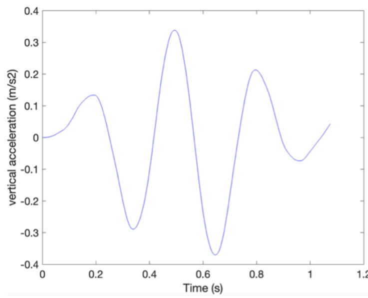

Figure 10: Time history of vertical acceleration of carbody calculated by current method

Figures 6 and 8 show respective displacements and accelerations respectively in bridge middle, while Figure 10 highlights accelerations in vehicle. The 3 curves are clearly obtained using current method. Figures 5, 7 and 9 report dynamic responses of railway system obtained using modal analysis method (MAM) presented in [8].

Calculated results adequately corroborate with those presented in [8], demonstrating validity and reliability of proposed method.

We note that the proposed approach is efficient and not expensive in terms of time. However, it is restricted to bridges of limited length, as it allows the assembly of only 10 elements of the rail-bridge sub-structure. In fact, in the finite element method, the smaller the elementary length, the greater the number of meshes, which also improves the accuracy of the calculations.

5. Conlcusion

Based on the energy approach and finite element method, motion equations of vehicle-rail-bridge system element have been established. The model presented divides the bridge and rail into 10 elements of equal length l. The vehicle is modeled by a suspended mass moving along the 10 elements at a constant speed v. The elements to be assembled are not identical which increases the complexity of their assembly. In fact, there are two types of elements to be assembled: one with a vehicle and 9 others without vehicle.

This study elaborates a procedure for assembling elementary matrices into the system global equations under study. These are then solved using the explicit Newmark beta numerical method to obtain the dynamic responses of overall vehicle-rail-bridge system.

The assembly procedure developed in this investigation is efficient, not expensive in terms of time and has been validated by comparing its results with those obtained by the method (MAM) presented in [8].

The method presented remains however restricted to bridges of limited length, since it only defines assembly of 10 elements. A more advanced technique for assembling n elements of elementary system is currently under investigation.

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